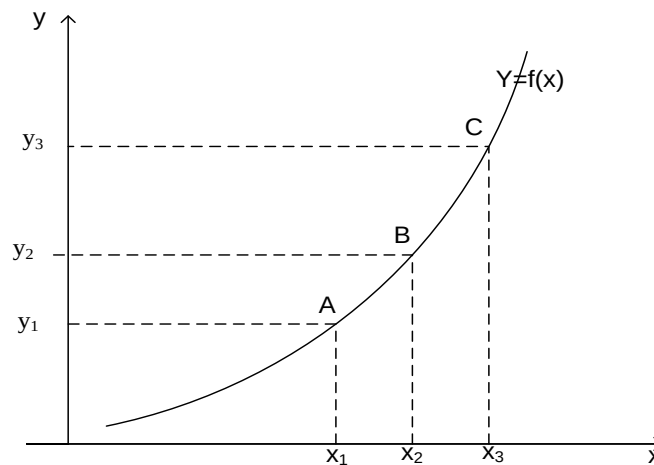


Topic 6: LINEAR INTERPOLATION AND EXTRAPOLATION

This is used to estimate values of a function by use of gradient method. If we are to locate a value which is not tabulated but lies within or outside two successive values x_1 and x_2 , we can use gradient approach to determine the unknown. Consider a function $y = f(x)$ with three close points; $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. If line AC is approximated to be a straight line then from the graph below the method of gradient can be used.



By equating correct gradients we get $\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

If the unknown point is B then, the method is referred to as linear interpolation and if the unknown point is A or C then we can use the same equation but the method now becomes linear extrapolation.

Example: The table below shows the delivery charges by a bus company as shown below

Mass(g)	200	400	600
Charges(shs)	700	1,200	3,000

Using linear interpolation or extrapolation, find the:

- Delivery charges of a parcel weighing 352g
- Mass of a parcel whose delivery charge is shs 3,300

Soln:

- Since the unknown point lies between two points the method is linear interpolation. It is always advisable to start with the unknown since it becomes easy to simplify.

200	352	400
700	x	1200

$$\frac{x - 700}{352 - 200} = \frac{1200 - 700}{400 - 200}$$

$x = \text{shs}1080$. The delivery charge is shs1080

- Here we shall use linear extrapolation. It is always advisable to start with the unknown since it becomes easy to simplify.

400	600	y
1,200	3,000	3,300

$$\frac{y - 400}{3,300 - 1,200} = \frac{600 - 400}{3,000 - 1,200}$$

$$y = 633\frac{1}{3}g$$

Example: The table below shows values of a function $f(x)$ as a set of points.

X	1.8	2.0	2.2	2.4
$f(x)$	0.532	0.484	0.436	0.384

Use linear interpolation to estimate the value of:

- $f(2.08)$
- x corresponding to $f(x)=0.5$

soln

-

2.0	2.08	2.2
-----	------	-----

0.484	y	0.436
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$$\frac{y - 0.484}{2.08 - 2.0} = \frac{0.436 - 0.484}{2.2 - 2.0} \quad y = 0.4648$$

b)

1.8	x	2.0
0.532	0.5	0.484

$$\frac{x - 1.8}{0.5 - 0.532} = \frac{2.0 - 1.8}{0.484 - 0.532} \quad x = 1.933$$

ASSIGNMENT 6.1.9

Linear interpolation and Extrapolation

1. Given the table below:

x	0	10	20	30
y	6.6	2.9	-0.1	-2.9

Use linear interpolation to find:

- a) y when x is 16
b) x when y is -1

2. The bus stages along Jinja-Kampala are 10km apart. An express bus travelling between two towns only stops at these stages, except in case of an emergency when it is permitted to stop at a point between two stages. The fares up to the first, second, third and fourth stage from Jinja are 1,100/=, 1,500/=, 1,850/= and 2,000/= respectively. On a certain day, a passenger paid to travel from Jinja to the fourth stage, but fell sick and had to be left at a health centre, 33km from Jinja.
- a) Given that he was refunded money for the part of the journey he had not travelled, find the approximate amount of money he received?
- b) Another person who had only 1,650/= was allowed to board, but would be left at a point worth his money, how far from Jinja was he left?

3. The table below is an extract from the table of $\cos x$

80°	0°	10'	20'	30'	40'	50'
-----	----	-----	-----	-----	-----	-----

$\cos x$	0.1736	0.1708	0.1679	0.1650	0.1622	0.1593
----------	--------	--------	--------	--------	--------	--------

Determine;

i) $\cos 80^{\circ} 36'$

ii) $\cos^{-1} 0.1685$

4. The table below shows values of $\sqrt{x}$ for given values of x .

x	0.12	0.13	0.14	0.15
$\sqrt{x}$	0.3464	0.3606	0.3742	0.3873

Using linear interpolation or extrapolation, find

a) $\sqrt{0.135}$

b) The value of x corresponding to $\sqrt{x} = 0.403$

TOPIC 8: ERROR ANALYSIS

An error is an inaccuracy which cannot be avoided in a measurement or a calculation. In math an error is not a mistake because a mistake can be avoided if one is careful. Errors are made during measurement of rainfall, atmospheric pressure, weights, and calculations using estimated values.

TYPES OF ERRORS

1. **Random errors.**

These occur due to human failure or due to machine failure. They cannot be treated numerically. Example; a student being given 54% instead of 34%

2. **Rounding errors.**

Some numbers are normally corrected to a given number of decimal places or significant figures.

Example: Round off:

- a) 3.896234 to 4 decimal places
= 3.8962
- b) 12.4872 to 2 decimal places
= 12.49
- c) 0.00652673 to 3 significant figures
= 0.00653
- d) 543216 to 3 significant figures
= 543216
- e) 546321 to 2 significant figures
= 550000

NOTE: Rounding off should be done once, i.e. it should be done in a single step

3. **Truncation errors.**

Occurs when an infinite process or value is terminated at some point.

Example: Truncate:

1. 0.66666667 to 4 significant figures
= 0.6666
2. 6.00513 to 2 decimal places
= 6.00

Note: Truncations can also be used to write an expansion e.g The expansion of $(1+x)^n$ to a given number of terms.

Common terms used in errors

1. Approximations:

An approximation is a value which is close to an exact value. E.g if the exact value is 4.321, then rounding off the value is an approximation.

2. Error :

An error is the difference between the exact value and the approximate value. It can be positive or negative.

Suppose X represents the exact value and x the corresponding approximate value then the error in x denoted by $\Delta x = X - x$. ie Error = exact – estimate

3. **Absolute error:** If Δx is the error in the estimate x , then the absolute error in x is $|\Delta x|$, disregarding the sign. $|\Delta x| = |X - x|$

4. Relative error:

If Δx is the error in x then the relative error in x is

$$\text{Relative error} = \frac{\text{absolute error}}{\text{exact or approximate value}}$$

$$= \left| \frac{\Delta x}{x} \right| \quad \text{or} \quad \left| \frac{\Delta x}{X} \right| \quad \text{since } X \approx x$$

5. Percentage error/ percentage relative error:

$$\text{Percentage error} = \left| \frac{\text{error}}{\text{estimate}} \times 100\% \right|$$

The percentage sign (%) can be neglected.

6. **The triangular inequality:** It is useful in deducing maximum errors. It is given by

$$|x + y| \leq |x| + |y|$$

Note:

1. The maximum error made in rounding off a number is given by $\text{Error} = \frac{1}{2} \times 10^{-n}$, where n is the number of decimal places to which the number is rounded off. This also gives the error when the number is rounded off to a given number of significant figures.

Example: Write down the maximum possible error in the following numbers correct to a given number of decimal places.

a) 2.31

$$\text{Error} = 0.005$$

b) 23.1

$$\text{Error} = 0.05$$

c) 0.0420

$$\text{Error} = 0.00005$$

7. Limits of accuracy:

These are maximum (upper) and minimum (lower) bounds of given figures.

Example: if $x = 5.53$ and the number is rounded off. Find the maximum and minimum values of x and state the interval in which the exact value of x lies.

Soln

$$\text{Maximum possible error in } x, \Delta x = 0.5 \times 10^{-2} = 0.005$$

$$\text{Minimum value} = x - \Delta x$$

$$= 5.53 - 0.005$$

$$= 5.525$$

$$\text{Maximum value} = x + \Delta x$$

$$= 5.53 + 0.005$$

$$= 5.535$$

$$\text{Interval/range} = [5.525, 5.535] \text{ or } 5.525 \leq x \leq 5.535$$

NOTE:

1. Always use closed brackets. Do not use open brackets
2. If the maximum and minimum values are known then we can also say that

$$\text{Maximum absolute error, } e = \frac{\text{maximum value} - \text{minimum value}}{2}$$

This implies that Error bound = $\pm e$

Maximum and minimum values of expressions

If x and y are approximate values with errors Δx and Δy respectively then we can find the maximum and minimum values of

a) $(x + y)$

$$(x + y)_{\max} = x_{\max} + y_{\max}$$

$$(x + y)_{\min} = x_{\min} + y_{\min}$$

b) $(x - y)$

$$(x - y)_{\max} = x_{\max} - y_{\min}$$

$$(x - y)_{\min} = x_{\min} - y_{\max}$$

c) $\frac{x}{y}$

$$\left(\frac{x}{y}\right)_{\max} = \frac{x_{\max}}{y_{\min}}$$

$$\left(\frac{x}{y}\right)_{\min} = \frac{x_{\min}}{y_{\max}}$$

Note: Special cases for quotients

$$\text{i) } \left(\frac{x}{x - y}\right)_{\max} = \frac{(x)_{\max}}{(x - y)_{\min}}, \left(\frac{x}{x - y}\right)_{\min} = \frac{(x)_{\min}}{(x - y)_{\max}} \text{ when } x > y \text{ (both}$$

numerator and denominator are positive)

$$\text{ii) } \left(\frac{x - y}{x + y}\right)_{\max} = \frac{(x - y)_{\max}}{(x + y)_{\max}}, \left(\frac{x - y}{x + y}\right)_{\min} = \frac{(x - y)_{\min}}{(x + y)_{\min}} \text{ when } y > x \text{ (the}$$

numerator is negative)

$$\text{iii) } \left(\frac{y}{x - y}\right)_{\max} = \frac{(y)_{\min}}{(x - y)_{\min}}, \left(\frac{y}{x - y}\right)_{\min} = \frac{(y)_{\max}}{(x - y)_{\max}} \text{ when } y > x \text{ (the}$$

denominator is negative)

d) (xy)

$$(xy)_{\max} = x_{\max} y_{\max}$$

$$(xy)_{\min} = x_{\min} y_{\min}$$

Example 1:

If $x=5.53$ and $y = 6.81$ and both numbers are rounded off.

a) State the maximum possible errors in x and y .

b) Find the:

I. Maximum value of $(x + y)$

II. Interval within which the exact value of $\frac{x}{y}$ lies.

Soln

a) $\Delta x = 0.005$ $\Delta y = 0.005$

b) i) $(x+y)_{\max} = x_{\max} + y_{\max}$
 $= (5.53+0.005) + (6.81+0.005)$
 $= 12.35$

ii) $\left(\frac{x}{y}\right)_{\max} = \frac{x_{\max}}{y_{\min}} = \frac{5.535}{6.805} = 0.813$

$\left(\frac{x}{y}\right)_{\min} = \frac{x_{\min}}{y_{\max}} = \frac{5.525}{6.815}$ hence the interval/range = $[0.811, 0.813]$

Example 2:

Given that $p = \frac{15.36 \times 27.1 - 1.672}{2.36 \times 1.043}$ the numbers are rounded off. Find:

i) The error in the calculation

ii) The value of the expression with error bounds

iii) The range within which the exact value lies

Soln

i) Error in $15.36 = 0.005$

Error in $27.1 = 0.05$

Error in $1.672 = 0.0005$

Error in $2.36 = 0.005$

Error in 1.043=0.0005

$$p_{\max} = \frac{15.365 \times 27.15 - 1.6715}{2.355 \times 1.0425}$$

$$= 169.2356179$$

$$p_{\min} = \frac{15.355 \times 27.05 - 1.6725}{2.365 \times 1.0435}$$

$$= 167.6259255$$

$$\begin{aligned} \text{Error in } P &= \frac{p_{\max} - p_{\min}}{2} \\ &= \frac{169.2356179 - 167.6259255}{2} \\ &= 0.8048462 \end{aligned}$$

$$\begin{aligned} \text{ii) Working value} &= \frac{15.36 \times 27.1 - 1.673}{2.36 \times 1.043} \\ &= 168.42875 \end{aligned}$$

Value with error bounds = 168.42875 ± 0.804846

$$\text{iii) Range} = [167.6259255, 169.2356179]$$

Example 3:

The sides of a rectangle are measured as 5.24cm and 6.38cm. Calculate the;

- Least value of the perimeter
- Limits within which the exact value of the area lies, hence determine the absolute error.

Soln



$$\text{i) Perimeter} = 2(l+w)$$

$$\text{Least value} = 2(6.375 + 5.235) = 23.22\text{cm}$$

$$\begin{aligned} \text{ii) Upper limit of area} &= 5.245 \times 6.385 \\ &= 33.489325\text{cm}^2 \end{aligned}$$

$$\text{Lower limit of area} = 5.235 \times 6.375 = 33.373125\text{cm}^2$$

$$\text{Absolute error} = \frac{\text{max} - \text{min}}{2} = \frac{33.489325 - 33.373125}{2} = 0.0581$$

Example 4:

The numbers $x = 27.23$, $y = 12.18$ and $z = 5.12$ are calculated with percentage errors of 4, 3 and 2 respectively. Find the minimum value of $xy - \frac{y}{z}$, correct to two decimal places.

Soln

$$\text{Percentage error in } x = \frac{\Delta x}{x} \times 100$$

$$4 = \frac{\Delta x}{27.23} \times 100$$

$$\Delta x = 1.0892$$

$$\Delta y = \frac{3 \times 12.18}{100}$$

$$\Delta y = 0.3654$$

$$\Delta z = \frac{2 \times 5.12}{100} = 0.1024$$

Then

$$\left(xy - \frac{y}{z} \right)_{\min} = (xy)_{\min} - \frac{y_{\max}}{z_{\min}}$$

$$= (27.23 - 1.0892)(12.18 - 0.3654) - \frac{(12.18 + 0.3654)}{(5.12 - 0.1024)} = 306.34$$

Deriving formula for error propagation

Suppose x and y are approximations of X and Y respectively. Let Δx and Δy be the corresponding errors in x and y respectively, then;

a) Error in $(x + y)$

$$\text{Exact value} = X + Y$$

$$= (x + \Delta x) + (y + \Delta y)$$

$$\text{Approximate value} = x + y$$

$$\text{Error in } x + y = (x + \Delta x) + (y + \Delta y) - (x + y)$$

$$= \Delta x + \Delta y$$

$$\text{Absolute error} = |\Delta x + \Delta y|$$

Since $|\Delta x + \Delta y| \leq |\Delta x| + |\Delta y|$, Therefore the maximum absolute error in $(x + y)$ is $|\Delta x| + |\Delta y|$

b) Error in $x - y$

$$\text{Exact value} = X - Y$$

$$= (x + \Delta x) - (y + \Delta y)$$

$$\text{Estimate value} = x - y$$

$$\text{Error in } x - y = (x + \Delta x) - (y + \Delta y) - (x - y)$$

$$= \Delta x - \Delta y$$

$$\text{Absolute error} = |\Delta x - \Delta y|$$

Since $|\Delta x - \Delta y| \leq |\Delta x| + |\Delta y|$, then the maximum absolute error in $x - y$ is $|\Delta x| + |\Delta y|$

Activity: Show that the maximum absolute error in $x + y$ is $\frac{|\Delta x| + \Delta y}{|x + y|}$

c) Error in xy

$$\text{Exact value} = XY$$

$$= (x + \Delta x)(y + \Delta y)$$

$$\text{Estimate value} = xy$$

$$\begin{aligned}\text{Error in } xy &= (x + \Delta x)(y + \Delta y) - xy \\ &= x\Delta y + y\Delta x + \Delta x\Delta y. \text{ For small } \Delta x, \Delta y, \Delta x\Delta y \approx 0\end{aligned}$$

(assumption)

$$\text{Error in } xy = x\Delta y + y\Delta x$$

$$\text{Absolute error in } xy = |x\Delta y + y\Delta x|. \text{ Since } |x\Delta y + y\Delta x| \leq |x\Delta y| + |y\Delta x|$$

Hence the maximum absolute error in xy is $|x\Delta y| + |y\Delta x|$

Note: From error in $xy = x\Delta y + y\Delta x$

$$\text{Absolute error in } xy = |x\Delta y + y\Delta x|$$

$$\text{Relative error} = \left| \frac{x\Delta y + y\Delta x}{xy} \right|. \text{ Since } \left| \frac{x\Delta y + y\Delta x}{xy} \right| \leq \left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|, \text{ the}$$

$$\text{maximum absolute relative error in } xy \text{ is } \left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|$$

Example 5: If x and y are approximations to X and Y with errors of Δx and Δy respectively, Show that;

i) The maximum absolute error in $\frac{x}{y}$ is given by $\frac{|y||\Delta x| + |x||\Delta y|}{y^2}$

ii) The maximum possible relative error in $\frac{x}{y}$ is given by $\left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|$

Soln

$$\text{Exact value} = \frac{X}{Y}$$

$$= \frac{x + \Delta x}{y + \Delta y}$$

$$\text{Estimate value} = \frac{x}{y}$$

$$\text{Error in } \frac{x}{y} = \frac{x + \Delta x}{y + \Delta y} - \frac{x}{y}$$

$$= \left(\frac{x + \Delta x}{y} \right) \left(1 + \frac{\Delta y}{y} \right)^{-1} - \frac{x}{y}$$

But from $\left(1 + \frac{\Delta y}{y}\right)^{-1} = 1 - \frac{\Delta y}{y} + \frac{\Delta y^2}{y^2} + \dots$

Assumption: For small Δy , $\Delta y^2 \approx 0$ and neglecting higher powers since they are very small too.

$$\left(1 + \frac{\Delta y}{y}\right)^{-1} = 1 - \frac{\Delta y}{y}$$

$$\begin{aligned} \text{Error in } \frac{x}{y} &= \left(\frac{x + \Delta x}{y}\right) \left(1 - \frac{\Delta y}{y}\right) - \frac{x}{y} \\ &= \frac{-x\Delta y}{y^2} + \frac{\Delta x}{y} - \frac{\Delta x\Delta y}{y^2} \text{ For small } \Delta x, \Delta y, \Delta x\Delta y \approx 0 \end{aligned}$$

$$\text{Error in } \frac{x}{y} = \frac{\Delta x}{y} - \frac{x\Delta y}{y^2}$$

Absolute error in $\frac{x}{y} = \left| \frac{y\Delta x - x\Delta y}{y^2} \right|$ since $\left| \frac{y\Delta x - x\Delta y}{y^2} \right| \leq \frac{|y\Delta x| + |x\Delta y|}{y^2}$. Therefore the

maximum absolute error in $\frac{x}{y}$ is $\frac{|y||\Delta x| + |x||\Delta y|}{y^2}$

$$\text{Relative error} = \left| \frac{y\Delta x - x\Delta y}{y^2} \right| \div \frac{x}{y} = \left| \frac{y\Delta x - x\Delta y}{xy} \right| = \left| \frac{\Delta x}{x} - \frac{\Delta y}{y} \right|$$

Since $\left| \frac{\Delta x}{x} - \frac{\Delta y}{y} \right| \leq \left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|$. Hence the maximum relative error in $\frac{x}{y}$ is $\left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|$

Note: Never use. As $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta x\Delta y \rightarrow 0$. When dealing with errors we are dealing with numbers. Instead we use $\Delta x\Delta y \approx 0$

Errors in functions $f(x)$

These include $y = \cos x$, $y = \sin x$, $y = 2^x$ and other trigonometric and exponential functions. Use of calculus can be used.

Consider $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ for small changes and since errors are small changes then

$$\frac{\Delta y}{\Delta x} \approx f'(x) \Delta y \approx \Delta x f'(x) \text{ but } \Delta y \text{ represents the error in } f(x)$$

Example:

Error in x^n

$$\text{Let } f(x) = x^n \Rightarrow f'(x) = n x^{n-1}$$

$$\text{Error in } f(x) = n \Delta x x^{n-1}$$

Alternatively: Suppose x is an approximate value of X and Δx is the error in x

$$\text{Exact value} = f(X)$$

$$= f(x + \Delta x)$$

$$\text{Estimate} = f(x)$$

$$\text{Error in } f(x) = f(x + \Delta x) - f(x)$$

Using Taylor's expansion

$$f(a + h) = f(a) + h f'(a) + \frac{h^2 f''(a)}{2!} + \dots \text{Where } h \text{ is very small compared to } a$$

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2 f''(x)}{2!} + \dots$$

For small Δx , $\Delta x^2 \approx 0$

$$f(x + \Delta x) = f(x) + \Delta x f'(x)$$

$$\text{Error in } f(x) = f(x) + \Delta x f'(x) - f(x)$$

$$= \Delta x f^1(x)$$

The absolute error in $f(x) = \left| \Delta x f^1(x) \right|$

Example 6:

Find the errors in the following functions.

- i) $\sin x$ ii) $\cos x$. Given that $x = 30^\circ$ and is rounded off.

Soln

- i) Let $f(x) = \sin x, x = 30^\circ, \Delta x = 0.5^\circ = \frac{0.5\pi}{180}$ in radians

Error in $f(x) = \Delta x f^1(x)$

$$\begin{aligned} &= \left| \Delta x \cos x \right| \\ &= \left| \frac{0.5}{180} \cos 30 \right| \\ &= 0.0075575 \end{aligned}$$

- ii) Let $f(x) = \cos x$

$$\begin{aligned} \text{Error} &= \left| \Delta x f^1(x) \right| \\ &= \left| -\Delta x \sin x \right| \\ &= \left| -\frac{0.5\pi}{180} \sin 30 \right| \\ &= 0.0043633 \end{aligned}$$

Note: For angles in degrees the error must be changed to radians.

Example 2:

If $y = \sin \theta$, find the interval within which y lies given that $\theta = 60^\circ$.

Soln

$$\Delta y = \left| \Delta \theta \cdot y^1(\theta) \right|$$

$$\Delta\theta = 0.5^\circ = 0.5 \frac{\pi}{180} \text{ radians}$$

$$\Delta y = \left| \frac{0.5\pi}{180} \cos 60 \right| = 0.004363$$

$$y_{\max} = y + \Delta y = \sin 60 + 0.004363 = 0.87039$$

$$y_{\min} = y - \Delta y = \sin 60 - 0.004363 = 0.8617$$

$$\text{Interval}[0.8617, 0.87039]$$

WORKED EXAMPLE:

Derive an expression for the maximum absolute relative error in x^2y with an estimate of x and y hence find the maximum percentage error in x^2y if $x = 3.14, y = 2.888$ and are rounded off.

Soln

$$\text{Error in } x^2y = (x + \Delta x)^2(y + \Delta y) - x^2y$$

$$= (x^2 + 2x\Delta x + \Delta x^2)(y + \Delta y) - x^2y$$

For small $\Delta x, \Delta x^2 \approx 0$

$$\text{Error in } x^2y = (x^2 + 2x\Delta x)(y + \Delta y) - x^2y$$

$$= x^2\Delta y + 2xy\Delta x + \Delta x\Delta y \text{ but } \Delta x\Delta y \approx 0$$

$$= x^2\Delta y + 2xy\Delta x$$

$$\text{Absolute relative error in } x^2y = \left| \frac{x^2\Delta y + 2xy\Delta x}{x^2y} \right| = \left| \frac{2\Delta x}{x} + \frac{\Delta y}{y} \right|$$

Since $\left| \frac{2\Delta x}{x} + \frac{\Delta y}{y} \right| \leq 2 \left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|$.

The maximum absolute relative error in x^2y is $2 \left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right|$

The percentage error in x^2y $= 2 \left| \frac{\Delta x}{x} \right| + \left| \frac{\Delta y}{y} \right| \times 100\%$

$$= \left\{ 2 \left(\frac{0.005}{3.14} \right) + \left(\frac{0.0005}{2.888} \right) \right\} \times 100$$

$$= 0.34\%$$

ASSIGNMENT 8.1.11

1. A value of $P=673.16$ was obtained in a certain experiment. Given that the relative error in the measurement of this value is 0.01% , find the limits within which the value of P is expected lie.
2. The relative error obtained in determining the value of $T=873.16$ is 0.02% , find
 - (i) The error in the measurement of this value
 - (ii) The value within which T lies
3. A student measured the length and the breadth of a rectangular sheet of iron as 3.6m and 2.3m respectively.
 - (i) Write down the maximum possible error in each measurement
 - (ii) Find the limits within which the area of the sheet lies.
4. Given that $Z = |x||y|\sin\theta$
 - (a) Derive an expression for the maximum possible relative error in Z is given that Δx , Δy and $\Delta\theta$ are small numbers compared to x , y and θ respectively

(b) Find the maximum percentage relative error in Z , given that $x = 5.5\text{cm}$, $y = 16.8\text{cm}$ and $\theta = 45^\circ$ and are rounded off.

5. Find the range of values within which the exact value of

$2.6954 \left(4.6006 - \frac{1.6175}{0.82} \right)$ lies if the numbers are rounded off to the given number of decimal places.

6. a) Given that $x = 4.00$ and $y = 2.0$, find the maximum error in $\frac{x+y}{x-y}$,

correct to 4 decimal places.

b) Given that $y = 5^x$ and x is measured with a value of 2.45, determine the absolute error in y hence Or otherwise determine the interval within which y lies. (Hint: Use error in a function)

7. (a) Round off to three significant figures;

(i) 6.9449 (ii) 10.459 (iii) 12436 (v) 0.01004

b)) Numbers X and Y were estimated with maximum possible errors of ΔX and ΔY respectively. Show that the maximum possible relative error in the estimation of $X\sqrt{Y}$ is given $\left| \frac{\Delta X}{X} \right| + \frac{1}{2} \left| \frac{\Delta Y}{Y} \right|$

c) Given that $A=7.4$, $B=80.03$ and $C=14.801$ are rounded off with corresponding percentage errors of 0.5, 0.5 and 0.005. Calculate the relative error in; i) $\frac{AB}{C}$

ii) $A-C$

8. (a) Given that a and b are estimated with corresponding errors of Δa and Δb . Show that the relative error in the product ab is $\left| \frac{\Delta a}{a} \right| + \left| \frac{\Delta b}{b} \right|$.

(b) The values $p = 4.7$, $q = 80.00$ and $r = 15.900$ are rounded off with corresponding percentage errors of 0.5, 0.05 and 0.05. Find the relative error in $\left(\frac{q}{r} - p \right)$.

9. a) Two sides of a triangle PQR are p and q such that $\angle PRQ = \alpha$.

i) Find the maximum possible error in the area of this triangle

ii) hence find the percentage error made in the area if $p = 4.5\text{cm}$,

$q = 8.4\text{cm}$ and $\alpha = 30^\circ$

(b) Find the range within which $\frac{3.679}{2} - \frac{7.0}{5.48}$ lies.

10. If $y = 5^{2x}$, find the absolute error in y when $x = 0.21$. (Hint: Use error in a function/calculus)

11. An error of 2.5% is made in measuring the area of a circle. Determine the corresponding percentage error in its radius.

12. Evaluate with error bounds $\sin 30^\circ$. (Hint: Use error in a function/calculus can be used)

13. If $x = 4.95$ and $y = 2.013$ are each rounded off to a given number of decimal places, calculate the maximum and minimum values of

i) $\frac{y-x}{x+y}$

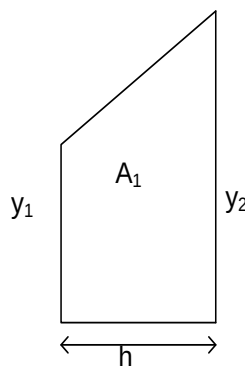
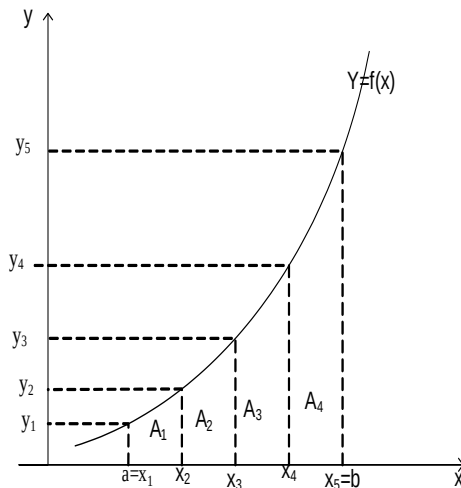
ii) $\frac{y^2}{y-x}$ (hint: special case for quotients)

Topic 7: Numerical integration

The trapezium rule: It is a numerical method used to estimate the area under a curve or to estimate the value of an integral where Calculus can't be applicable.

Note: Accurate area under the curve or a value of an integral is obtained by calculus, i.e integration.

Therefore the error in the value obtained by trapezium rule = value obtained by calculus - value by trapezium rule. Consider a function $y = f(x)$, if the area under the curve is divided into equal trapezoids each of width, h



$$\text{Total area} = \int_a^b f(x)dx = A_1 + A_2 + A_3 + A_4$$

$$A_1 = \frac{1}{2} h(y_1 + y_2)$$

$$A_2 = \frac{1}{2} h(y_2 + y_3)$$

$$A_3 = \frac{1}{2} h(y_3 + y_4)$$

$$A_4 = \frac{1}{2} h(y_4 + y_5)$$

$$\text{Then the total area} = \frac{1}{2} h(y_1 + 2(y_2 + y_3 + y_4) + y_5)$$

$$= \frac{1}{2} h\{y_1 + y_5 + 2(y_2 + y_3 + y_4)\}$$

Hence the general trapezium rule formula can be written as

$$\int_a^b f(x)dx = \frac{1}{2} h\{y_1 + y_n + 2(y_2 + y_3 + y_4 + \dots + y_{n-1})\}$$

Note:

1. The areas A_1, A_2, A_3 , and A_4 are called strips or sub-intervals
2. The x-values, x_1, x_2, x_3, x_4 and x_5 are called ordinates.
 $x_1 = a, x_2 = a + h, x_3 = x_2 + h$
3. Number of subintervals = number of ordinates $- 1$
4. $h = \frac{b-a}{n}$, where n = number of subintervals or strips.
5. h must remain as a fraction if it does not give an exact decimal
6. When dealing with trigonometric functions, leave the calculator in radians

Example 1:

a) Use trapezium rule with 5 subintervals to estimate $\int_0^1 5^{2x} dx$, and give your answer correct to 3 decimal places.

b) i) Find the exact value of the $\int_0^1 5^{2x} dx$ correct to 3 decimal places

ii) Calculate the percentage error in your estimation in a) above

iii) Suggest how the percentage error may be reduced
soln

Note: y values must be atleast 4 decimal places since the final answer is required to 3 decimal places

a) $h = \frac{1-0}{5} = 0.2$

x	y
0	1
0.2	1.9036
0.4	3.6239
0.6	6.8986
0.8	13.1327
1	25
Sub-totals	26 51.1176

Then $\int_0^1 5^{2x} dx = \frac{1}{2} (0.2) \{26 + 2(51.1176)\}$

$= 7.71176$

≈ 7.712

$$b) i) \int_0^1 5^{2x} dx = \left[\frac{5^{2x}}{2 \ln 5} \right]_0^1 = \frac{5^2}{2 \ln 5} - \frac{1}{2 \ln 5} = 7.456$$

$$\text{Error} = |7.456 - 7.712| = 0.256$$

$$ii) \text{Percentage error} = \frac{0.256}{7.456} \times 100 = 3.4\%$$

iii) Percentage error can be reduced by increasing the number of subintervals

Example:

Use trapezium rule with 6 ordinates to estimate $\int_0^{\frac{\pi}{3}} \tan x dx$ correct to 3 decimal places

soln

$$h = \frac{\frac{\pi}{3} - 0}{6 - 1} = \frac{\pi}{15}$$

x	y
0	0
$\frac{\pi}{15}$	0.2126
$\frac{2\pi}{15}$	0.4452
$\frac{3\pi}{15}$	0.7263
$\frac{4\pi}{15}$	1.1106
$\frac{\pi}{3}$	1.7321
Sub-totals	1.7321 2.4947

$$\int_0^{\frac{\pi}{3}} \tan x dx = \frac{1}{2} \left(\frac{\pi}{15} \right) \{1.732 + 2(2.4947)\} = 0.703873 \approx 0.704$$

ASSIGNMENT 7.1.10

1. (a) Use trapezium rule with six strips to estimate $\int_0^{\pi} x \sin x dx$ correct to 2dp.
 (b) Determine the percentage relative error in your estimation.

2. Use trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+x^2} dx$ using 6 ordinates correct to 3 decimal places.

3. (a) Use the trapezium rule with six ordinates to evaluate, $\int_0^1 \theta \sin \theta d\theta$ correct to three decimal places.
 (b) Find the actual value of, correct to three decimal places.
 (c) Calculate the relative error made in (a) above and state how you would reduce such an error.

4. Use trapezium rule with 6- ordinates to estimate the value of $\int_0^{\frac{\pi}{4}} (t + \sin t) dt$ correct to three decimal places.

TOPIC 9: ESTIMATION OF ROOTS OF EQUATIONS

Some equations of the form $f(x) = 0$ have no exact method of being solved. Various methods have been designed to find approximate solutions to the equations.

9.1 Methods of locating roots of equations:

a) Graphical method:

In graphical method of locating roots of the equation $f(x) = 0$, we first write it in the form $h(x) = g(x)$. The graphs of $y = g(x)$ and $y = h(x)$ are plotted or sketched on the same axes. The values of x where the graphs meet will be the point where the root lies and it's the approximate root of the equation. Alternatively: The graph of $y = f(x)$ is plotted or sketched. The point where the root cuts the x -axis will be the approximate root of the equation.

Example 1:

Show graphically the root of the equation $x^3 + 5x^2 - 3x - 4 = 0$, a root between $x = 0$ and $x = -1$.

Soln:

Method 1

Let $f(x) = x^3 + 5x^2 - 3x - 4$

x	-1	-0.8	-0.6	-0.4	-0.2	0
y	3	1.1	-0.6	-2.1	-3.2	-4

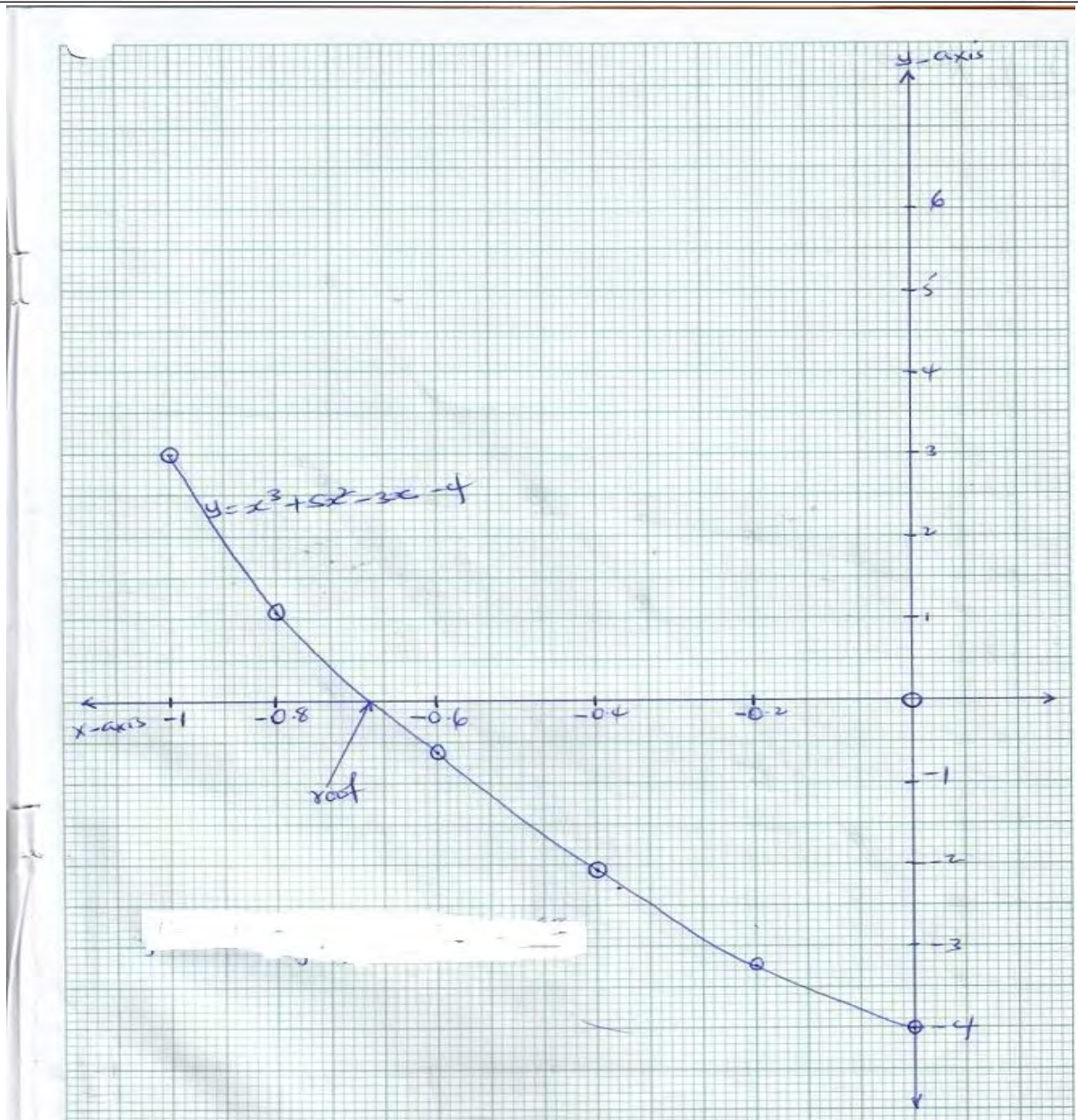
From the graph figure, below $x_0 = -0.68$

Method 2

The equation above can be separated into $y_1 = x^3 + 5x^2$, and, $y_2 = 3x + 4$

The two graphs are plotted the two points where the graphs meet will be the approximate root of the equation, $x^3 + 5x^2 - 3x - 4 = 0$

Note: When plotting a curve a minimum of four points is required.



Example 2:

Using a suitable graph locate roots of the equation $x^3 + x - 4 = 0$

Soln:

Method I

Let $y = x^3 + x - 4$

x	-3	-2	-1	0	1	2	3
y	-34	-14	-6	-4	-2	6	26

b) The sign change method

The sign change method of locating the root of the equation $f(x) = 0$ between a and b. There exists a root between a and b if and only if $f(a)$ and $f(b)$ have opposite signs ie the function changes through a and b

Example:

Show that the equation $x^3 + 2x - 4 = 0$ has a root between $x = 1$ and $x = 2$

$$f(x) = x^3 + 2x - 4$$

$$f(1) = 1^3 + 2(1) - 4 = -1$$

$$f(2) = 2^3 + 2(2) - 4 = 8$$

Since $f(1).f(2) < 0$ then there exists a root between $x = 1$ and $x = 2$

OR Since $f(1)$ and $f(2)$ have different signs then there exists a root between $x = 1$ and $x = 2$

Example:

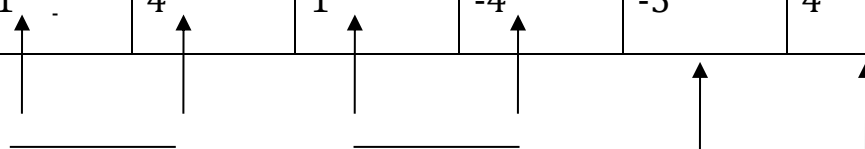
Show that $x^3 - 4x^2 + 4 = 0$ has one negative root and two positive roots.

Soln:

Since the interval is not given we can use a table

Let $f(x) = x^3 - 4x^2 + 4$

X	-1	0	1	2	3	4
f(x)	-1	4	1	-4	-5	4



Sign changes

Since $f(-1) \cdot f(0) < 0$, $f(1) \cdot f(2) < 0$ and $f(3) \cdot f(4) < 0$. This implies that the roots of the equation above lie between -1 and 0; 1 and 2; and 3 and 4.

9.2 Finding an approximate root

This can be done using two methods

- i) By graphical method (already discussed)
- ii) Linear interpolation

Linear interpolation method:

If b is an approximate root of the equation $f(x) = 0$ then $f(b) = 0$. If $a < b < c$, then

a	b	c
+ value	0	- value

OR

a	b	c
- value	0	+ value

Example 3:

Show that the equation $3x^2 + x - 5 = 0$ has a root between $x=1$ and $x=2$ hence use linear interpolation to determine the **initial approximation**.

Soln

Let $f(x) = 3x^2 + x - 5$

$$f(1) = 3(1)^2 + 1 - 5 = -1$$

$f(2) = 3(2)^2 + 2 - 5 = 9$ Since $f(1) \cdot f(2) < 0$ then there exists a root between $x=1$ and $x=2$

Note: For an approximation interpolation is only used once

Then; let the initial approximation, $b = x_0$

x	1	x_0	2
f(x)	-1	0	9

$$\frac{x_0 - 1}{0 - (-1)} = \frac{2 - 1}{9 - (-1)}$$

$$x_0 = 1.1$$

9.3: Finding the root of the equation

The methods include;

1. Linear interpolation method
2. Iteration methods

Linear interpolation method: Here the linear interpolation method is used more than once until $|f(b)| \leq 0.5 \times 10^{-n}$ where n is the number of decimal places required.

Example 4: Show that the equation $3x^2 + x - 5 = 0$ has a root between $x=1$ and $x=1.5$ hence use linear interpolation to determine the root of the equation correct to 2 decimal places.

Soln

Since the accurate root is required linear interpolation should be done more than once.

Let $f(x) = 3x^2 + x - 5$

$$f(1) = 3(1)^2 + 1 - 5 = -1$$

$f(2) = 3(1.5)^2 + 1.5 - 5 = 3.25$ Since $f(1) \cdot f(1.5) < 0$ then there exists a root between $x=1$ and $x=1.5$

Since $x=1.5$ gives a value far 0 then another value e.g 1.25 (average of 1 and 1.5) can be used

$$f(1.25) = 3(1.25)^2 + 1.25 - 5 = 0.9375$$

Either (a)

1	x_0	1.25
-1	0	3.25

OR (b) it is also ok to use 1 and 1.5

1	x_0	1.5
-1	0	9

From a) $\frac{x_0 - 1}{0 - -1} = \frac{1.25 - 1}{0.9375 - -1}, x_0 = 1.12903$

Testing:

$$f(1.12903) = 3 \times 1.12903^2 + 1.12903 - 5 = -0.0468$$

Since $f(1.12903) \cdot f(1.25) < 0$ then

1.12903	x_1	1.25
-0.0468	0	0.9375

$$\frac{x_1 - 1.12903}{0 - -0.0468} = \frac{1.25 - 1.12903}{0.9375 - -0.0468}$$

$$x_1 = 1.13478$$

Testing:

$$f(1.13478) = 3(1.13478)^2 + 1.13478 - 5 = -0.00204$$

$$\text{Since } |f(1.13478)| = 0.00204 < 0.005$$

$$\therefore \text{Root} = 1.13$$

ITERATIVE METHODS OF SOLVING $f(x) = 0$

Iteration means logical steps. An iterative method is therefore a numerical method which is step by step of obtaining an estimate to a value of the root such that the successive approximations from the sequence converge to a value of the root. The methods include:

- i) the general iterative method
- ii) the Newton Raphson iterative method

General iterative method

Here the iterative formula to be used in estimating the root of the equation is obtained by rearranging the terms of the equation and written in the form

$$X_{n+1} = g(x_n) \text{ Where } n=0, 1, 2, 3, \dots$$

Example 5: Given the equation $x^3 + x - 5 = 0$. We can generate a number of formulas as shown.

$$\begin{aligned} \text{a)} \quad & x^3 + x - 5 = 0 \\ & x = 5 - x^3 \\ & X_{n+1} = 5 - x^3 \end{aligned}$$

b) $x^3 = 5 - x$
 $x = \sqrt[3]{5 - x}$
 $X_{n+1} = \sqrt[3]{5 - x_n}$

c) $x^2 + 1 - \frac{5}{x} = 0$
 $x = \sqrt{\frac{5}{x} - 1}$
 $X_{n+1} = \sqrt{\frac{5}{x_n} - 1}$

Note: Not all the iterative formulas obtained by rearrangements of terms are suitable to estimate a root therefore test for suitability of a general iterative formula

Test for convergence

Mehod 1

Example 6:

The iterative formulas below are formed by rearranging the equation

$$x^3 - 10 = 0$$

a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ and b) $X_{n+1} = \frac{10}{x_n^2}$. Starting with an approximation of

$x_0 = 2$ use each formula three times to determine the most suitable formula for solving the equation, hence state the root of the equation correct to significant figures.

Solution:

Formula a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$

Formula b) $X_{n+1} = \frac{10}{x_n^2}$

$$X_1 = \frac{2(2)^3 + 10}{3(2)^2} = 2.1667$$

$$X_2 = \frac{2(2.1667)^3 + 10}{3(2.1667)^2} = 2.1544$$

$$X_3 = \frac{2(2.1544)^3 + 10}{3(2.1544)^2} = 2.1544$$

$$X_1 = \frac{10}{2^2} = 2.5$$

$$X_2 = \frac{10}{2.5^2} = 1.6$$

$$X_3 = \frac{10}{1.6^2} = 3.9063$$

Formula a) is the suitable formula because it produces a convergent sequence hence the root is 2.154.

Note: Substitution at every stage must be seen

Method 2

Second test for suitability of an iterative formula. If $|g^1(x_0)| < 1$, it implies that the iterative formula is suitable otherwise not

Example 7: The iterative formulas below are formed by rearranging the equation $x^3 - 10 = 0$

a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ and b) $X_{n+1} = \frac{10}{x_n^2}$. Taking $x_0 = 2$, **without iteration**,

deduce the most suitable formula for solving the equation.

Solution:

Formula a) $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$

$$\Rightarrow g(x) = \frac{2x^3 + 10}{3x^2} = \frac{2x}{3} + \frac{10}{3x^2}$$

$$g^1(x) = \frac{2}{3} - \frac{20}{3x^3}$$

$$g^1(2) = \frac{2}{3} - \frac{20}{24} = \frac{1}{6}$$

$$|g^1(2)| = \frac{1}{6} < 1$$

Formula b) $X_{n+1} = \frac{10}{x_n^2}$

$$\Rightarrow g(x) = \frac{10}{x^2}$$

$$g^1(x) = \frac{-20}{x^3}$$

$$|g^1(2)| = \left| \frac{-20}{2^3} \right| = 2.5 > 1$$

Since $|g^1(2)| < 1$ in a) then formula a) given by $X_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$ is the more suitable formula for solving the equation.

Example 8: Show that the general iterative formula for solving the equation $x^3 - x - 1 = 0$ is $X_{n+1} = \sqrt{1 + \frac{1}{x_n}}$. Hence taking 1.2 as the first approximation, find the root of the equation correct to 2 decimal places

Solution:

$$x^3 - x - 1 = 0$$

$$x^3 = x + 1$$

$$x^2 = 1 + \frac{1}{x}$$

$$x = \sqrt{1 + \frac{1}{x}}$$

$$X_{n+1} = \sqrt{1 + \frac{1}{x_n}}$$

$$x_1 = \sqrt{1 + \frac{1}{1.2}} = 1.354$$

$$|x_1 - x_0| = 0.154$$

$$x_2 = \sqrt{1 + \frac{1}{1.354}} = 1.3185$$

$$|x_2 - x_1| = 0.0355$$

$$x_3 = \sqrt{1 + \frac{1}{1.3185}} = 1.3261$$

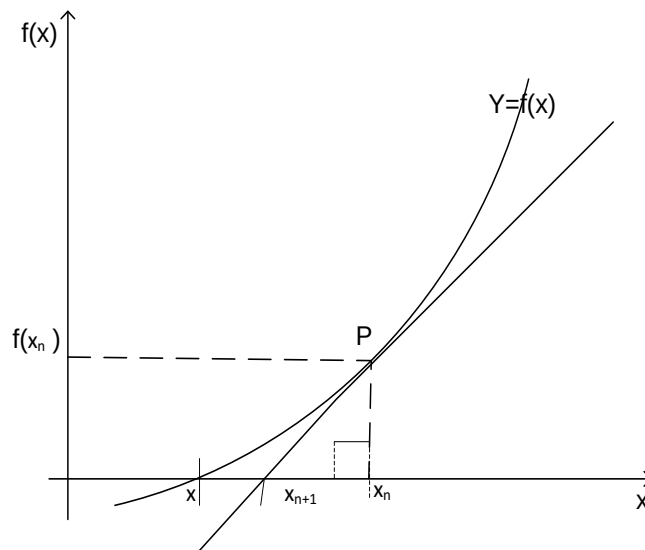
$$|x_3 - x_2| = 0.0076$$

$$x_4 = \sqrt{1 + \frac{1}{1.3261}} = 1.3244$$

$$|x_4 - x_3| = 0.0017 \leq 0.005, \text{ Hence the root is } 1.32$$

The Newton Raphson method (N.R.M)

Consider an equation $f(x) = 0$, when the graph is plotted it cuts the x-axis at a point $(x, 0)$ where x is the root of the equation. A tangent at $P[x_n, f(x_n)]$ is drawn where x_n is an approximation to the root. The tangent meets the x-axis at $(x_{n+1}, 0)$, where x_{n+1} is a better approximation since it is closer to x . Graphically we have



$f(x)=0$, and x_n is the initial approximation to the root. The gradient of the tangent at P is $f'(x_n)$ and at the same time as $\tan \theta$ where

$$\tan \theta = \frac{f(x_n) - 0}{x_n - x_{n+1}} = f'(x_n)$$

Rearranging the equation above we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} ; n = 0, 1, 2, 3, \dots \text{ This expression is the Newton}$$

Raphson formula

Example 9:

a) Show that the Newton Raphson formula for solving the equation

$$2x^2 - 6x - 3 = 0 \text{ is } x_{n+1} = \frac{2x_n^2 + 3}{4x_n - 6}, \text{ Where } n = 0, 1, 2, \dots$$

- b) Show that the positive root for $2x^2 - 6x - 3 = 0$ lies between 3 and 4.
Find the root correct to 2 decimal places

Solution

a) Let $f(x) = 2x^2 - 6x - 3$

$$f'(x) = 4x - 6$$

Then $X_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$X_{n+1} = x_n - \frac{2x_n^2 - 6x_n - 3}{4x_n - 6}$$

$$X_{n+1} = \frac{4x_n^2 - 6x_n - 2x_n^2 + 6x_n + 3}{4x_n - 6}$$

$$\therefore X_{n+1} = \frac{2x_n^2 + 3}{4x_n - 6}$$

$$f(3) = 2(3)^2 - 6 \times 3 - 3 = -3$$

b)

$$f(4) = 2(4)^2 - 6 \times 4 - 3 = 5$$

Since $f(4) \cdot f(3) < 0$ then there exists a root between 3 and 4

Since $|f(3)| < |f(4)|$ then $x_0 = 3$

$$x_1 = \frac{2(3)^2 + 3}{4(3) - 6} = 3.5$$

$$x_2 = \frac{2(3.5)^2 + 3}{4(3.5) - 6} = 3.4375$$

$$x_3 = \frac{2(3.4375)^2 + 3}{4(3.4375) - 6} = 3.4365$$

$$|x_3 - x_2| = |3.4365 - 3.4375| = 0.002 < 0.005$$

$$\text{Root} = 3.44$$

Example 10:

Show that the Newton –Raphson Iterative formula for finding the cube root of a number N is given by $X_{n+1} = \frac{1}{3} \left(2x_n + \frac{N}{x_n^2} \right); n = 0, 1, 2, \dots$ hence find the $(96)^{\frac{1}{3}}$ correct to 3 decimal places

Solution

$$\text{Let } x = \sqrt[3]{N}$$

$$x^3 = N$$

$$x^3 - N = 0$$

$$f(x) = x^3 - N$$

$$f'(x) = 3x^2$$

$$X_{n+1} = x_n - \left(\frac{x_n^3 - N}{3x_n^2} \right)$$

$$= \frac{3x_n^3 - x_n^3 - N}{3x_n^2} = \frac{1}{3} \left(2x_n + \frac{N}{x_n^2} \right); \text{ where } n=0, 1, 2, \dots$$

$$X_{n+1} = \frac{1}{3} \left(2x_n + \frac{N}{x_n^2} \right); \quad n=0, 1, 2, \dots$$

$$\sqrt[3]{96} = ?$$

$$\sqrt[3]{64} = 4 < \sqrt[3]{96} = ? < \sqrt[3]{125} = 5$$

$$f(x) = x^3 - N$$

$$f(4) = 4^3 - 96 = -32$$

$$f(5) = 5^3 - 96 = 29$$

$$\text{Since, } |f(5)| < |f(4)|, \text{ then, } x_0 = 5$$

$$x_1 = \frac{1}{3} \left(2(5) + \frac{96}{5^2} \right) = 4.6133$$

$$x_2 = \frac{1}{3} \left(2(4.6133) + \frac{96}{4.6133^2} \right) = 4.57911$$

$$x_3 = \frac{1}{3} \left(2(4.57911) + \frac{96}{4.57911^2} \right) = 4.5789$$

$$|x_3 - x_2| = 0.00021 < 0.0005,$$

$$\therefore \text{Root} = 4.579$$

Example 11:

- a) Draw on the same axes the graphs of $y = 2 - e^{-x}$ and $y = \sqrt{x}$ for $2 \leq x \leq 5$
- b) Determine from your graphs the interval within which the root of the equation $e^{-x} + \sqrt{x} - 2 = 0$ lies. Hence, use Newton Raphson's method to find the root of the equation correct to 3 decimal places.

Solution:

Let $y_1 = 2 - e^{-x}$ and $y_2 = \sqrt{x}$

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y_1	1.86	1.92	1.95	1.97	1.98	1.99	1.99
y_2	1.41	1.58	1.73	1.87	2.00	2.12	2.24

From the graph, the roots of the equation lies between $x=3.5$ and $x=4.0$.

For the hence part:

$$f(x) = e^{-x} + \sqrt{x} - 2, \Rightarrow f'(x) = e^{-x} + \frac{1}{2\sqrt{x}} = \frac{2e^{-x}\sqrt{x} + 1}{2\sqrt{x}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, 3, \dots$$

$$x_{n+1} = x_n - \frac{(e^{-x_n} + \sqrt{x_n} - 2) \times \sqrt{x_n}}{2e^{-x_n}\sqrt{x_n} + 1}$$

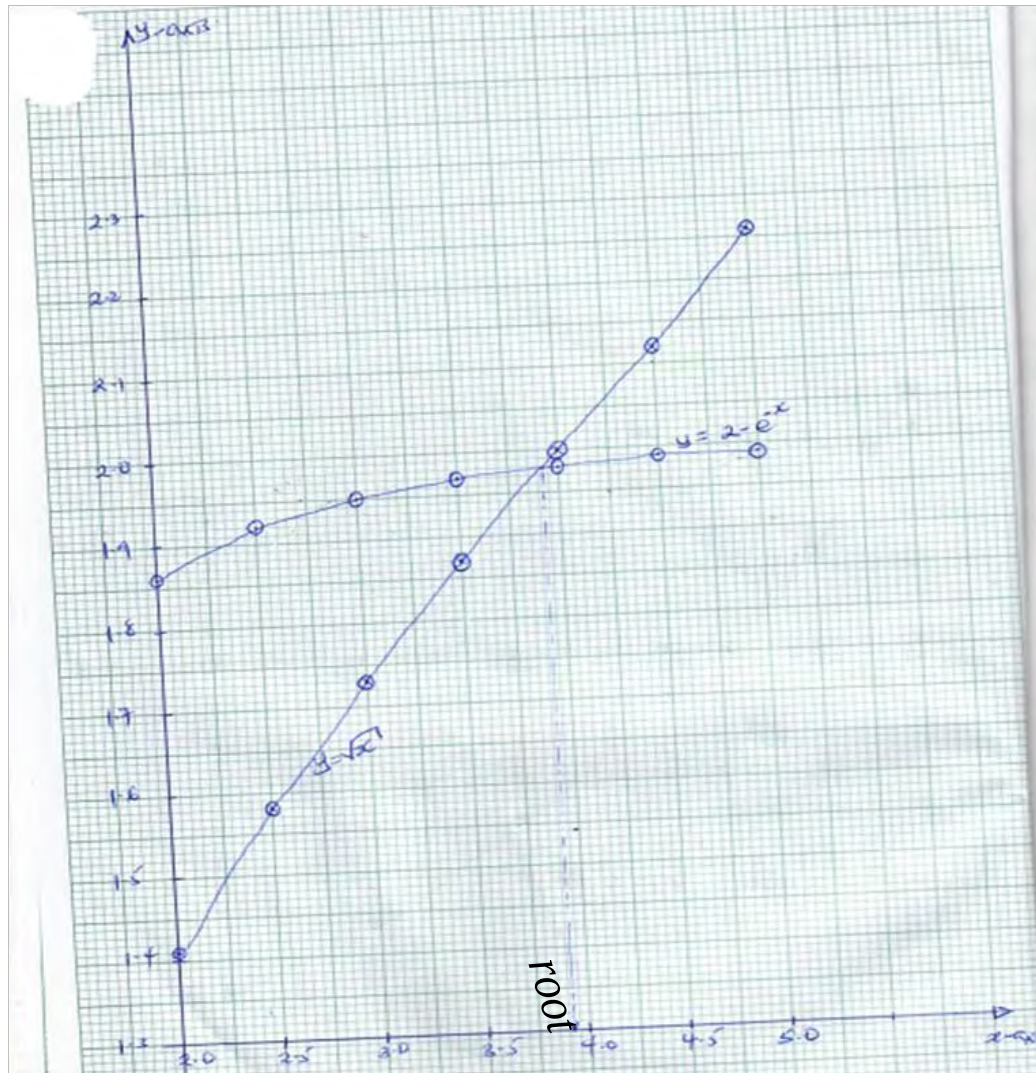
The initial approximation of the root $x_0 = 3.925$

$$x_1 = 3.925 - \frac{(e^{-3.925} + \sqrt{3.925} - 2) \times 2\sqrt{3.925}}{2e^{-3.925} \times \sqrt{3.925} + 1} = 3.92168$$

$$x_2 = 3.92168 - \frac{(e^{-3.92168} + \sqrt{3.92168 - 2}) \times 2\sqrt{3.92168}}{2e^{-3.92168} \times \sqrt{3.92168} + 1} = 3.9212$$

$$|x_2 - x_1| = |3.9212 - 3.92168| = 0.00048 < 0.0005$$

$$\therefore \text{root} = 3.921$$



Note:

- The axes must be labeled
- The graphs must be labeled

- Use of uniform scale

ASSIGNMENT 9.1.12

- Using the same graph, Show that the curves $2e^x$ and $4 - x^2$ have two real roots.
 - Using the Newton – Raphson formula thrice, find the positive root of the equation $2e^x + x^2 = 4$, giving your answer correct to two decimal places.
- Show that the equation $2x^3 - \frac{18}{x} + 2 = 0$ has two real roots in the interval $|x| < 3$. Use linear interpolation to find the least root , correct to one decimal place.
- Derive the iterative formula based on Newton- Raphson process for obtaining the reciprocal of a number N.
 - Hence, using N-R process once, and taking x_0 to be 0.1, find the approximate value of $\frac{1}{11}$, correct to 2 significant figures.
- Locate graphically the smallest positive real root of $\sin x - \ln x = 0$
 - Use Newton Raphson method to approximate this root of the equation in a) above correct to 3 dps.
- By plotting the graphs of $y = \ln x$ and $y = 2 - x$, Show that there is a common root in the interval $1 < x < 2$, obtain from your graph the approximate root of the equation $\ln x + x - 2 = 0$, correct to one decimal place.

6. Show graphically that the equation $\pi \sin x = x$ has three real roots.
(hint: Plot graphs of $y = \sin x$ and $y = \frac{x}{\pi}$. Use intervals of π)
7. If α is an approximate root of the equation $x^2 = n$. Show that the iterative formula for finding the root reduces to: $\frac{\frac{n}{\alpha} + \alpha}{2}$ hence, taking $\alpha = 4$, estimate $\sqrt{17}$, correct to 3 decimal places.
8. (a) Show that the root of the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ lies between 1 and 2.
(b) Use the Newton Raphson method to find the root of the equation above correct to 2 dps.
9. (a) If a is an approximate root of the equation $x^5 - b = 0$, Show the second approximation is given by $\frac{4a + \frac{b}{a^4}}{5}$
b) Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. hence use the formula in a) above to determine the root to three decimal places